Discrete Wavelet Transformations and Undergraduate Education

Catherine Bénéteau and Patrick J. Van Fleet

Wavelet theory was an immensely popular research area in the 1990s that synthesized ideas from mathematics, physics, electrical engineering, and computer science. In mathematics, the subject attracted researchers from areas such as real and harmonic analysis, statistics, and approximation theory, among others. Applications of wavelets abound today—perhaps the most significant contributions of wavelets can be found in signal processing and digital image compression. As the basic tenets of wavelet theory were established, they became part of graduate school courses and programs, but it is only in the last ten years that we have seen wavelets and their applications being introduced into the undergraduate curriculum.

The introduction of the topic to undergraduates is quite timely—most of the foundational questions posed by wavelet researchers have been answered, and several current applications of wavelets are firmly entrenched in areas of image processing. Several authors [7, 2, 8, 14] have written books in which they present the basic results of wavelet theory in a manner that is accessible to undergraduates. Others have authored books [15, 13, 1, 9] that include applications as part of their presentation.

In much the same way that wavelet theory is the confluence of several mathematical disciplines, we have discovered that the discrete wavelet transformation is an ideal topic for a modern undergraduate class—the derivation of the discrete wavelet transformation draws largely from calculus and linear algebra, provides a natural conduit to Fourier series and discrete convolution, and allows near-immediate access to current applications. Students learn about signal denoising, edge detection in digital images, and image compression, and use computer software in both the derivation and implementation of the discrete wavelet transformation. In the process of investigating applications, students learn how the application often drives the development of mathematical tools. Finally, the design of more advanced wavelet filters allows the students to gain experience "working in the transform domain" and provides motivation for several important concepts from an undergraduate real analysis class.

In what follows, we outline the basic development of discrete wavelet transformations and discuss their connection with Fourier series, convolution, and filtering. In the process, we illustrate the application of discrete wavelet transformations to o1ir

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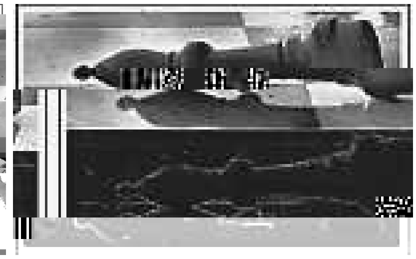
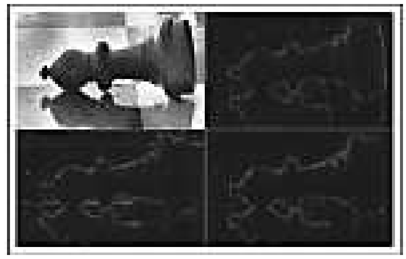
(a) A

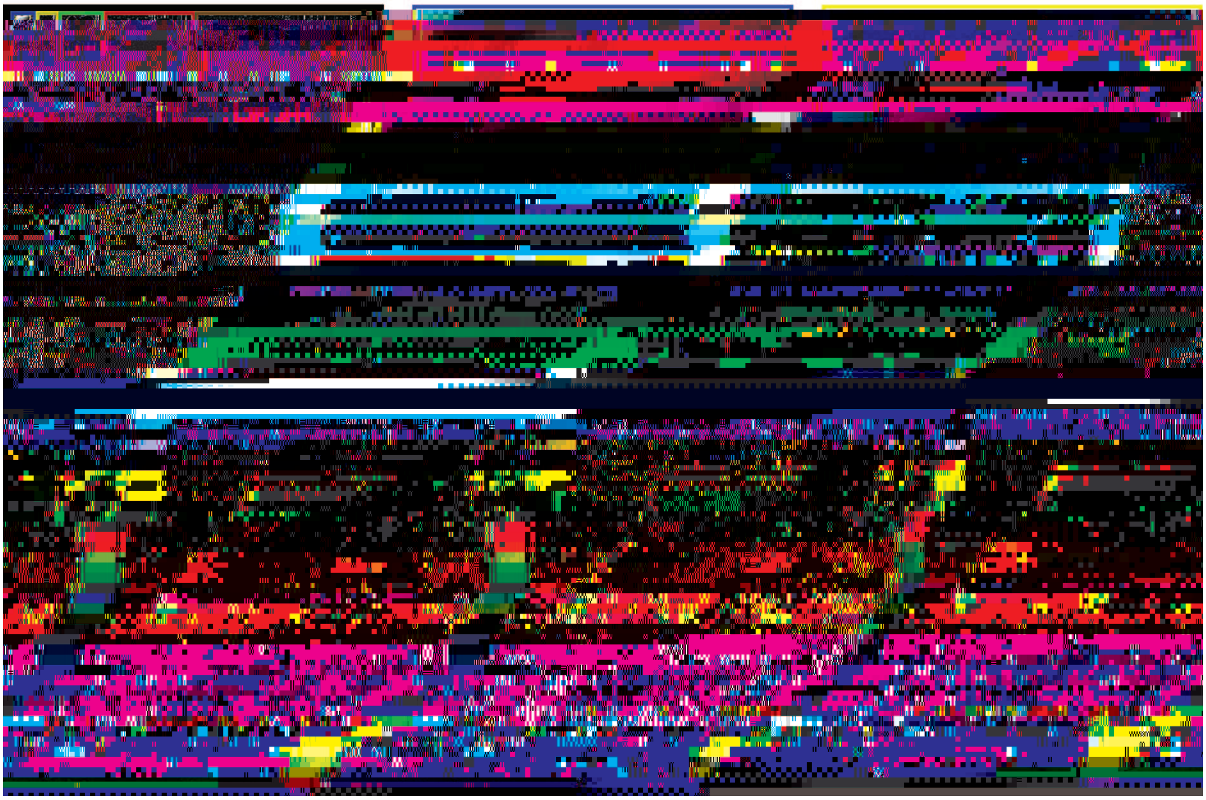


(b) $\tilde{W}_{320}A$



(c) $\tilde{W}_{320}A\tilde{W}_{512}^T$

$\tilde{W}_M A$.



Four Canoes

$\mathcal{B}_4$

$\mathcal{B}_4$, the fourth iteration averages portion of the transform, could serve as a thumbnail image for the original. If a user requests the original image, we could first transmit $\mathcal{B}_4$ and then progressively send detail portions at each iterative level. The recipient can apply the inverse transform as detail portions are received to sequentially produce a higher-resolution version of the original image. Figure 3 illustrates the process.

Equation (2) tells us that the discrete Haar wavelet transformation is almost orthogonal. Indeed, if we define $W_N = \sqrt{2}\tilde{W}_N$, then $W_N^T = W_N^{-1}$,
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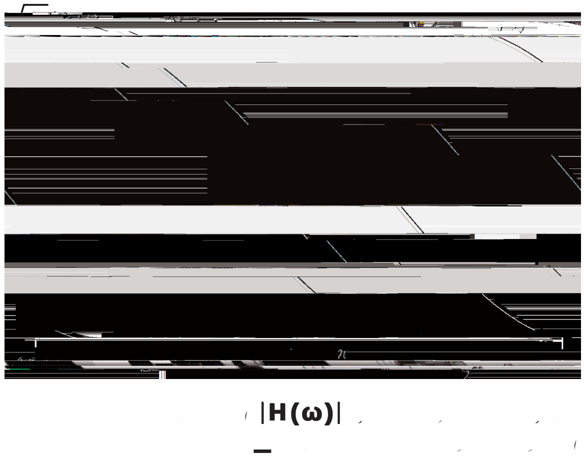
$\mathbf{y}$ is a vector whose elements are $y_k = (-1)^k$, $k \in \mathbb{Z}$. Then $\mathbf{h} * \mathbf{x} = \sqrt{2}\,\mathbf{x}$ and $\mathbf{h} * \mathbf{y}$ is the bi-infinite zero vector. Thus $\mathbf{h}$ is an example of a lowpass filter—it passes low-frequency signals through largely unchanged but attenuates the amplitudes of high-frequency data. In a similar manner we see that $\mathbf{g} * \mathbf{x}$ is the bi-infinite zero vector and $\mathbf{g} * \mathbf{y} = \sqrt{2}\mathbf{y}$. Here $\mathbf{g}$ is an example of a highpass filter—it allows high-frequency signals to pass largely unchanged but attenuates the amplitudes of low-frequency data. Thus in filtering terms, the discrete Haar wavelet transform is constructed by applying a lowpass filter and a highpass filter to the input data, downsampling both results, and then appropriately truncating the downsampled vectors. We will learn that all discrete wavelet transforms are built from a lowpass (scaling) filter $\mathbf{h}$ and a highpass (wavelet) filter. Fourier series are valuable tools for constructing these filters.

A course on discrete wavelet transformations is h **and** g



for $n \in \mathbb{Z}$, and

$$A(\omega)\overline{B(\omega)} + A(\omega+\pi)\overline{B(\omega+\pi)} = 0 \tag{20}$$

if and only if

$$\sum_{k\in\mathbb{Z}} a_k b_{k-2n} = 0 \tag{21}$$

for all $n \in \mathbb{Z}$.

The proof of this theorem is straightforward. Note that equations (19) and (21) guarantee orthogonality of the matrix W_N.

Suppose $H(\omega)$ satisfies (18). If we take

$$G(\omega) = -e^{iL\omega}\overline{H(\omega+\pi)}, \tag{22}$$

then $G(\omega)$ satisfies (18), and $H(\omega)$, $G(\omega)$ satisfy (20). It is an easy exercise to show that the Fourier coe cients of $G(\omega)$ are $g_k = (-1)^k h_{1-k}$, $k \in \mathbb{Z}$.

The condition $H'(\pi) = 0$ leads naturally to the following generalization: If we want to produce longer scaling filters $\mathbf{h}$, each time we increase the filter length by two, we require an additional derivative condition at π. For example, the Daubechies scaling filter of lengthisf

Fortunately, there is a way to deal with the wrapping row problem, and that is to develop filters that are symmetric.

An odd-length filter $\mathbf{h}$ is called symmetric if $h_k = h_{-k}$, while an even-length filter is called symmetric if $h_k = h_{1-k}$. Daubechies [5] proved that the only symmetric, finite length, orthogonal filter is the Haar filter, and we have already discussed some of its limitations. How can we produce symmetric filters while preserving some of the desirable properties of orthogonal filters? These desirable properties are finite length (computational speed), orthogonality (ease of inverse), and the ability to produce a good approximation of the original data with the scaling filter. Since the inverse need only be computed once, it was Daubechies's idea to relinquish orthogonality and to construct instead a discrete biorthogonal wavelet transformation.

The idea is to construct two sets of filters instead of one. In other words, we construct two wavelet transform matrices $\tilde{W}_N$ and W_N so that $\tilde{W}$

insist that both scaling filters be symmetric. Let's look at an example.

Example 1. Suppose we want $\tilde{h}$ to be a symmetric, length three, lowpass filter $\tilde{h} = [\tilde{h}_{-1}, \tilde{h}_0, \tilde{h}_1]^T = [\tilde{h}_1, \tilde{h}_0, \tilde{h}_1]^T$. We seek two numbers $\tilde{h}_0$, $\tilde{h}_1$ so that $\tilde{H}(0) = \tilde{h}_1 + \tilde{h}_0 + \tilde{h}_1 = \sqrt{2}$ and $\tilde{H}(\pi) = -\tilde{h}_1 + \tilde{h}_0 - \tilde{h}_1 = 0$. The filter[1] $\tilde{h}$ we seek is

$$[\tilde{h}_{-1}, \tilde{h}_0, \tilde{h}_1]^T = \frac{\sqrt{2}}{4}[1, 2, 1]^T.$$

To find a symmetric filter h, we must satisfy the biorthogonality condition (25), together with the lowpass constraints $H(0) = \sqrt{2}$ and $H(\pi) = 0$. It turns out that h must have an odd length of $5, 9, 13$

elements of a / (d) are built from the odd (even) elements of v